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Output agreement in networks with unmatched disturbances and algebraic constraints

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Claudio De Persis

Abstract—This paper considers a problem of output agreement in heterogeneous networks with dynamics on the nodes as well as on the edges. The control and disturbance signals entering the nodal dynamics are “unmatched” meaning that some nodes are only subject to disturbances, and are deprived of actuating signals. To further enrich our model, we accommodate (solvable) algebraic constraints in a subset of nodal dynamics. We show that appropriate dynamic feedback controllers achieve output agreement on a desired vector. We also investigate the case of an optimal steady-state control over the network. The proposed results are applied to a heterogeneous microgrid.

I. INTRODUCTION

Agreement on a certain quantity of interest plays a central role in cooperative control. The most notable instances are distributed optimization [16], consensus [13], formation control [12], and synchronization, see e.g. [14], [11], [15].

The study of output agreement/regulation problem in the presence of disturbances has been motivated by numerous applications in balancing demand and supply, power networks, and hydraulic networks. In this framework, the demands/loads are interpreted as external disturbances affecting the network dynamics, see e.g. [6], [4], [7].

In this paper, we consider agents with non-identical nonlinear port-Hamiltonian dynamics; see [18] for more information on port-Hamiltonian systems. The nodal dynamics is subject to constant disturbances. In addition, we consider that a subset of nodal dynamics is governed by algebraic constraints. These constraints could be the result of mismatch in the dynamic order of the agents, or an approximation of fast subdynamics in singularly perturbed models [10]. The algebraic constraints we consider here are solvable meaning that they can be expressed in terms of other state variables of the network. However, obviously, the presence of such constraints adds to the heterogeneity of the network, and complicates the analysis.

We consider the physical coupling to be “undamped”, and given by a single integrator with a nonlinear output map. We first show that an equilibrium of the network, if exists, is attractive and thus output agreement is locally achieved for the network. Next, we include controller dynamics on the nodes to guarantee output agreement on a prescribed set point, in the presence of physical coupling and disturbance signals. Another important feature here is that we treat an *unmatched* control-disturbance scheme, meaning that control signals and disturbances may act on different subsets of nodes. As a

case study, we consider a heterogeneous microgrid consisting of synchronous generators, droop-controlled inverters, and frequency dependent loads, where the goal is to guarantee a zero frequency deviation for all the nodes of the grid.

Note that the main contribution of the current manuscript is to consider simultaneously i) multivariable nonlinear nodal dynamics, ii) dynamic physical coupling, iii) algebraic constraints, and iv) unmatched disturbances in the output agreement problem. Our analysis is implicitly based on passivity and incremental passivity property inspired by [1], [2], [5], [3], [17].

The analysis of output agreement problem is carried out in Section II, whereas the control design is treated in Section III. Section IV is devoted to the case study of microgrids. Conclusions are provided in Section V. The formal proofs of the proposed results are collected in the appendix in Section VI.

Notation Apart from the standard notation, we use the following conventional notation. We use superscripts for vectors and matrices to indicate their domain of definition. In particular, let x_j with $j \in \mathcal{I}$ be a set of vectors. Then, by x^i we mean $x^i = \text{col}(x_j)$ with $j \in \mathcal{I}_i \subseteq \mathcal{I}$. For a set of matrices, we define $A^i = \text{blockdiag}(A_j)$ with $j \in \mathcal{I}_i \subseteq \mathcal{I}$. We remove the superscript in case $\mathcal{I}_i = \mathcal{I}$.

II. ANALYSIS

We define a dynamical network on a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We partition the vertex set of \mathcal{G} into two distinct subsets, $\mathcal{V} := \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. To each vertex of \mathcal{G} , we associate the following port-Hamiltonian types of dynamics:

$$\dot{x}_i = (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + d_i) \quad i \in \mathcal{I}_1 \quad (1a)$$

$$0 = (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + d_i) \quad i \in \mathcal{I}_2 \quad (1b)$$

$$y_i = G_i^T \nabla H_{n,i}(x_i) \quad i \in \mathcal{I} \quad (1c)$$

where $x_i \in \mathbb{R}^n$, J_i is a skew symmetric matrix, R_i is a positive definite matrix, $\sigma_i \in \mathbb{R}^m$ amounts for the physical coupling, $d_i \in \mathbb{R}^m$ is a constant vector, and the Hamiltonian $H_{n,i} : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly convex in an open convex set $\Omega_n \subseteq \mathbb{R}^n$ for each i .

To each edge of \mathcal{G} , we associate the following dynamics:

$$\dot{\eta}_k = v_k \quad (2a)$$

$$\mu_k = \nabla H_{e,k}(\eta_k) \quad (2b)$$

where $\eta_k \in \mathbb{R}^m$, the Hamiltonian $H_{e,k} : \mathbb{R}^m \rightarrow \mathbb{R}$ is strictly convex in an open convex set $\Omega_e \subseteq \mathbb{R}^m$, and

$k = 1, 2, \dots, M$. The interconnection law is given by

$$v = (B^T \otimes I_m)y, \quad \sigma = -(B \otimes I_m)\mu \quad (3)$$

where B is the incidence matrix of \mathcal{G} , $v = \text{col}(v_k)$, $y = \text{col}(y_i)$, and $\sigma = \text{col}(\sigma_i)$ with $k = 1, 2, \dots, M$ and $i = 1, 2, \dots, N$.

Then, the edge dynamics (2), the nodal dynamics (1), and the interconnection law (3) can be written compactly as

$$\dot{\eta} = (B^T \otimes I)G^T \nabla H_n(x) \quad (4a)$$

$$\mu = \nabla H_e(\eta) \quad (4b)$$

$$\begin{aligned} \dot{x}^1 &= (J^1 - R^1) \nabla H_n^1(x^1) \\ &\quad - G^1(B^1 \otimes I) \nabla H_e(\eta) + G^1 d^1 \end{aligned} \quad (4c)$$

$$\begin{aligned} 0 &= (J^2 - R^2) \nabla H_n^2(x^2) \\ &\quad - G^2(B^2 \otimes I) \nabla H_e(\eta) + G^2 d^2 \end{aligned} \quad (4d)$$

$$y = G^T \nabla H_n(x) \quad (4e)$$

where B^1 and B^2 denote the submatrices obtained from B by collecting the rows indexed by \mathcal{I}_1 and \mathcal{I}_2 , respectively. Let $x = \text{col}(x^1, x^2)$ and $d = \text{col}(d^1, d^2)$. Suppose that $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ is an equilibrium of system (4) with $\dot{\bar{x}} = 0$ and $\dot{\bar{\eta}} = 0$. Then, we have

$$0 = (B^T \otimes I)G^T \nabla H_n(\bar{x}) \quad (5a)$$

$$\begin{aligned} 0 &= (J^1 - R^1) \nabla H_n^1(\bar{x}^1) \\ &\quad - G^1(B^1 \otimes I) \nabla H_e(\bar{\eta}) + G^1 d^1 \end{aligned} \quad (5b)$$

$$\begin{aligned} 0 &= (J^2 - R^2) \nabla H_n^2(\bar{x}^2) \\ &\quad - G^2(B^2 \otimes I) \nabla H_e(\bar{\eta}) + G^2 d^2. \end{aligned} \quad (5c)$$

Observe that the equation (5a) yields an output agreement condition

$$G_i^T \nabla H_{n,i}(\bar{x}_i) = G_j^T \nabla H_{n,j}(\bar{x}_j), \quad \forall i, j \in \mathcal{I}. \quad (6)$$

Hence, we obtain that $G^T \nabla H_n(\bar{x}) = \mathbb{1}_N \otimes y^*$ for some constant vector $y^* \in \mathbb{R}^n$. The other two equations can be written together as

$$0 = (J - R) \nabla H_n(\bar{x}) - G(B \otimes I) \nabla H_e(\bar{\eta}) + Gd. \quad (7)$$

This implies that

$$\mathbb{1}_N \otimes y^* = G^T (J - R)^{-1} G ((B \otimes I) \nabla H_e(\bar{\eta}) - d). \quad (8)$$

In case the matrix G is equal to the identity matrix, by multiplying both hand sides of (8) from the left by $(\mathbb{1}_N^T \otimes I_n)(J - R)$, we obtain that

$$\sum_{i=1}^N (J_i - R_i) y^* = - \sum_{i=1}^N d_i. \quad (9)$$

Hence, $y^* = \nabla H_{n,i}(\bar{x}_i)$ is computed as

$$y^* = - \left(\sum_{i=1}^N (J_i - R_i) \right)^{-1} \sum_{i=1}^N d_i. \quad (10)$$

Then, noting that $\mathbb{1}_N \otimes y^* = \nabla H_n(\bar{x})$, the constant vector $\bar{x} \in (\Omega_n)^N$ is unique in this case. It is worth mentioning that

in the case $n = 1$, we have $J = 0$, and (10) is simplified to $y^* = \frac{\mathbb{1}^T d}{\mathbb{1}^T R \mathbb{1}}$.

By replacing (10) in (8) with $G = I$, the term $(B \otimes I) \nabla H_e(\bar{\eta})$ is explicitly computed. Hence, $\bar{\eta} \in (\Omega_e)^M$ in general is not unique. However, in case the graph \mathcal{G} is acyclic, the incidence matrix B has full column rank, and thus $\bar{\eta}$ is unique. Note that an equilibrium $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ does not always exist, and in particular the feasibility conditions (6) and (7) must hold. The following theorem investigates stability/attractivity properties of such an equilibrium, assuming that it exists.

Theorem 1 Suppose that $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ is an equilibrium of (4). Then there exists a region of state space, which includes $(\bar{x}, \bar{\eta})$, such that any solution (x, η) of (4) starting in this region asymptotically converges to an equilibrium of (4), and the output agreement condition (6) holds.

Proof. See Appendix.

Note that Theorem 1 implies that the network (4) reaches an output agreement providing that there exist constant vectors $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ satisfying (6), (7), and thus (8). As the vector y^* resulting from this agreement may be not the desired one, due to the dependency on the disturbance d , next we investigate the possibility to influence this vector by an appropriate control scheme.

III. CONTROL

In this section, we treat certain control problems related to network dynamics (4). To capture the heterogeneous role of the nodes, we further partition the nodal dynamics (1) as

$$\begin{aligned} \dot{x}_i &= (J_i - R_i) \nabla H_{n,i}(x_i) + G_i(\sigma_i + u_i + \delta_i) & i \in \mathcal{I}_{11} \\ \dot{x}_i &= (J_i - R_i) \nabla H_{n,i}(x_i) + G_i(\sigma_i + \delta_i) & i \in \mathcal{I}_{12} \\ 0 &= (J_i - R_i) \nabla H_{n,i}(x_i) + G_i(\sigma_i + u_i + \delta_i) & i \in \mathcal{I}_{21} \\ 0 &= (J_i - R_i) \nabla H_{n,i}(x_i) + G_i(\sigma_i + \delta_i) & i \in \mathcal{I}_{22} \\ y_i &= G_i^T \nabla H_{n,i}(x_i) & i \in \mathcal{I} \end{aligned} \quad (11)$$

where $\mathcal{I}_1 = \mathcal{I}_{11} \cup \mathcal{I}_{12}$, $\mathcal{I}_2 = \mathcal{I}_{21} \cup \mathcal{I}_{22}$, $\mathcal{I}_{11} \neq \emptyset$, and G_i has a full column rank for each i . Here, the u_i components are treated as control signals which are applied to the nodes, and the δ_i s are constant disturbance signals affecting the nodal dynamics. Note that the nodes in \mathcal{I}_{12} and \mathcal{I}_{22} are not directly controlled, and therefore our treatment here incorporates the case of an *unmatched* control-disturbance scheme.

Now, the overall network dynamics can be written as

$$\dot{\eta} = (B^T \otimes I)G^T \nabla H_n(x) \quad (12a)$$

$$\begin{aligned} \dot{x}^{11} &= (J^{11} - R^{11}) \nabla H_n^{11}(x^{11}) \\ &\quad - G^{11}(B^{11} \otimes I) \nabla H_e(\eta) \\ &\quad + G^{11}u^{11} + G^{11}\delta^{11} \end{aligned} \quad (12b)$$

$$\begin{aligned} \dot{x}^{12} &= (J^{12} - R^{12}) \nabla H_n^{12}(x^{12}) \\ &\quad - G^{12}(B^{12} \otimes I) \nabla H_e(\eta) + G^{12}\delta^{12} \end{aligned} \quad (12c)$$

$$0 = (J^{21} - R^{21}) \nabla H_n^{21}(x^{21})$$

$$\begin{aligned}
& -G^{21}(B^{21} \otimes I)\nabla H_e(\eta) \\
& + G^{21}u^{21} + G^{21}\delta^{21}
\end{aligned} \tag{12d}$$

$$\begin{aligned}
0 = (J^{22} - R^{22})\nabla H_n^{22}(\bar{x}^{22}) \\
- G^{22}(B^{22} \otimes I)\nabla H_e(\eta) + G^{22}\delta^{22}
\end{aligned} \tag{12e}$$

$$y = G^T \nabla H_n(x). \tag{12f}$$

Our goal here is to design dynamic feedback controllers u^{11} and u^{21} such that output agreement (6) is guaranteed for the network, for a prescribed vector y^* , in the presence of network coupling and disturbance signals. If such u^{11} and u^{21} exist, we say that the output agreement problem is *solvable*. Obviously, this may not be always plausible, and by (12) we obtain the following feasibility condition

$$\mathbf{1} \otimes y^* = G^T \nabla H_n(\bar{x}) \tag{13a}$$

$$\begin{aligned}
0 = (J^{11} - R^{11})\nabla H_n^{11}(\bar{x}^{11}) \\
- G^{11}(B^{11} \otimes I)\nabla H_e(\bar{\eta}) \\
+ G^{11}\bar{u}^{11} + G^{11}\delta^{11}
\end{aligned} \tag{13b}$$

$$\begin{aligned}
0 = (J^{12} - R^{12})\nabla H_n^{12}(\bar{x}^{12}) \\
- G^{12}(B^{12} \otimes I)\nabla H_e(\bar{\eta}) + G^{12}\delta^{12}
\end{aligned} \tag{13c}$$

$$\begin{aligned}
0 = (J^{21} - R^{21})\nabla H_n^{21}(\bar{x}^{21}) \\
- G^{21}(B^{21} \otimes I)\nabla H_e(\bar{\eta}) \\
+ G^{21}\bar{u}^{21} + G^{21}\delta^{21}
\end{aligned} \tag{13d}$$

$$\begin{aligned}
0 = (J^{22} - R^{22})\nabla H_n^{22}(\bar{x}^{22}) \\
- G^{22}(B^{22} \otimes I)\nabla H_e(\bar{\eta}) + G^{22}\delta^{22}
\end{aligned} \tag{13e}$$

Clearly, this boils down to the following condition.

Feasibility condition: there exist constant vectors $\bar{x} \in (\Omega_n)^N$, $\bar{\eta} \in (\Omega_e)^M$, d^{11} , d^{21} such that

$$\mathbf{1} \otimes y^* = G^T \nabla H_n(\bar{x}) \tag{14a}$$

$$\begin{aligned}
0 = (J^{11} - R^{11})\nabla H_n^{11}(\bar{x}^{11}) \\
- G^{11}(B^{11} \otimes I)\nabla H_e(\bar{\eta}) + G^{11}d^{11}
\end{aligned} \tag{14b}$$

$$\begin{aligned}
0 = (J^{12} - R^{12})\nabla H_n^{12}(\bar{x}^{12}) \\
- G^{12}(B^{12} \otimes I)\nabla H_e(\bar{\eta}) + G^{12}\delta^{12}
\end{aligned} \tag{14c}$$

$$\begin{aligned}
0 = (J^{21} - R^{21})\nabla H_n^{21}(\bar{x}^{21}) \\
- G^{21}(B^{21} \otimes I)\nabla H_e(\bar{\eta}) + G^{21}d^{21}
\end{aligned} \tag{14d}$$

$$\begin{aligned}
0 = (J^{22} - R^{22})\nabla H_n^{22}(\bar{x}^{22}) \\
- G^{22}(B^{22} \otimes I)\nabla H_e(\bar{\eta}) + G^{22}\delta^{22}
\end{aligned} \tag{14e}$$

Note that we have used the fact that $\bar{\eta}$ is constant, and G^{11} and G^{22} are full column rank. Now, we have the following result.

Theorem 2 Consider the decentralized controller

$$\dot{\xi}_i = y^* - G_i^T \nabla H_{n,i}(x_i) \tag{15a}$$

$$u_i = \xi_i \tag{15b}$$

with $i \in \mathcal{I}_{11} \cup \mathcal{I}_{21}$. Assume that the feasibility condition (14) holds, and let $\xi = \text{col}(\xi_i)$ and $\bar{\xi} = \text{col}(d^{11}, d^{21})$. Then, there exists a region of state space, including $(\bar{x}, \bar{\eta}, \bar{\xi})$, such that any solution (x, η, ξ) of the network asymptotically converges to an equilibrium of (12) and (15), in which $G_i^T \nabla H_{n,i}(\bar{x}_i) = y^*$ for each $i \in \mathcal{V}$.

Proof. See Appendix.

Remark 3 Note that in case the controller at a node $i \in \mathcal{I}_{11}$ or $i \in \mathcal{I}_{21}$ does not have access to the desired output y^* , one can set u_i to a constant, namely a nominal value, and incorporate the node i in the subdynamics of (11) corresponding to the nodes indexed by \mathcal{I}_{12} or \mathcal{I}_{22} , respectively.

In Theorem 2, the control input u has been designed such that output agreement on a prescribed vector y^* is achieved for the network. Observe that the “steady-state” control signal $\bar{u} = \bar{\xi}$ is primarily determined by the initialization of the system/controller. Next, under the constraint of output agreement (6), we aim to minimize the following quadratic cost function

$$\min_{\bar{u}} = \frac{1}{2} \sum_{i \in \mathcal{I}_c} \bar{u}_i^T Q_i \bar{u}_i \tag{16}$$

where $Q_i \in \mathbb{R}^m \times \mathbb{R}^m$ is a positive definite matrix for each i , and $\mathcal{I}_c = \mathcal{I}_{11} \cup \mathcal{I}_{21}$. Note that the optimization above determines the steady-state distribution of the control effort over the agents of the network. This is particularly relevant in applications involving demand and supply balancing, including power as well as hydraulic networks; see e.g. [4], [6].

To make the analysis more concise, we restrict our attention to the case where $G_i = I$ for each i . Then, similar to (9), we obtain the following constraint

$$\sum_{i=1}^N (J_i - R_i)y^* + \sum_{i \in \mathcal{I}_c} u_i + \sum_{i=1}^N \delta_i = 0. \tag{17}$$

By standard Lagrange multipliers method, the vector \bar{u} which minimizes (16) subject to (17) is obtained as

$$\bar{u}_i = Q_i^{-1} \lambda \tag{18}$$

where $\lambda \in \mathbb{R}^n$ is given by

$$\lambda = -\left(\sum_{i \in \mathcal{I}_c} Q_i^{-1}\right)^{-1} \left(\sum_{i=1}^N (J_i - R_i)y^* + \sum_{i=1}^N \delta_i\right). \tag{19}$$

It is easy to observe that, by (13) and (18), we obtain the following feasibility condition in this case.

Feasibility condition with optimality: For a given $y^* \in \Omega_n$, there exists a constant vector $\bar{\eta} \in (\Omega_e)^M$ such that

$$\begin{aligned}
0 = (J^{11} - R^{11})(\mathbf{1} \otimes y^*) - (B^{11} \otimes I)\nabla H_e(\bar{\eta}) \\
+ (Q^{11})^{-1}(\mathbf{1} \otimes \lambda) + \delta^{11}
\end{aligned} \tag{20a}$$

$$\begin{aligned}
0 = (J^{12} - R^{12})(\mathbf{1} \otimes y^*) - (B^{12} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{12}
\end{aligned} \tag{20b}$$

$$\begin{aligned}
0 = (J^{21} - R^{21})(\mathbf{1} \otimes y^*) - (B^{21} \otimes I)\nabla H_e(\bar{\eta}) \\
+ (Q^{21})^{-1}(\mathbf{1} \otimes \lambda) + \delta^{21}
\end{aligned} \tag{20c}$$

$$\begin{aligned}
0 = (J^{22} - R^{22})(\mathbf{1} \otimes y^*) - (B^{22} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{22}
\end{aligned} \tag{20d}$$

where λ is as in (19).

To achieve output agreement problem with an optimal “steady state” control input, we propose a distributed controller at the nodes. The communication among the controllers takes place over a communication graph, say $\mathcal{G}_C = (\mathcal{V}_c, \mathcal{E}_c)$, which is undirected and connected.

Theorem 4 Assume that the feasibility condition (20) holds. Consider the distributed controller

$$\dot{\xi}_i = \sum_{\{i,j\} \in \mathcal{E}_c} (\xi_j - \xi_i) + Q_i^{-1}(y^* - \nabla H_{n,i}(x_i)) \quad (21a)$$

$$u_i = Q_i^{-1} \xi_i \quad (21b)$$

with $i \in \mathcal{I}_{11} \cup \mathcal{I}_{21}$. Let $\xi = \text{col}(\xi_i)$, and let the constant vector $\bar{\xi}$ be chosen as $\bar{\xi} = \mathbf{1} \otimes \lambda$ where λ is given by (19). Then, there exists a region of state space, including $(\bar{x}, \bar{\eta}, \bar{\xi})$, such that any solution (x, η, ξ) of the network starting in this region asymptotically converges to an equilibrium of (12) and (21), in which $\nabla H_{n,i}(\bar{x}_i) = y^*$ for each $i \in \mathcal{V}$. Moreover, in this region, u_i asymptotically converges to the optimal \bar{u}_i given by (18).

Proof. See Appendix.

IV. CASE STUDY

We consider a (fairly) general heterogeneous microgrid which consists of synchronous generators, droop-controlled inverters, and frequency dependent loads. We partition the buses, i.e. the nodes of \mathcal{G} , into three sets, namely \mathcal{V}_G , \mathcal{V}_I , and \mathcal{V}_L , corresponding to the set of synchronous generators, inverters, and loads, respectively.

The dynamics of each synchronous generator is governed by the so-called *swing equation*, and is given by:

$$M_i \ddot{\theta}_i = -A_i \dot{\theta}_i + u_i - P_i + \delta_i, \quad i \in \mathcal{V}_G, \quad (22)$$

where

$$P_i = \sum_{\{i,j\} \in \mathcal{E}} \text{Im}(Y_{ij}) V_i V_j \sin(\theta_i - \theta_j) \quad (23)$$

is the active nodal injection at node i . Here, $M_i > 0$ is the moment of inertia, $A_i > 0$ is the damping constant, u_i is the local controllable power generation, and δ_i is the local load at node $i \in \mathcal{V}_G$. The value of $Y_{ij} \in \mathbb{C}$ is equal to the admittance of the branch $\{i, j\} \in \mathcal{E}$, and θ_i is the voltage angle at node i . Also, V_i is the voltage magnitude at node i , and is assumed to be constant.

For the droop-controlled inverters, we consider the following first-order model

$$A_i \dot{\theta}_i = u_i - P_i + \delta_i, \quad i \in \mathcal{V}_I \quad (24)$$

where A_i is known as the droop coefficient, u_i is the injection power at node (inverter) i , δ_i is the local load at inverter i , and $\dot{\theta}_i$ indicates the frequency deviation from the nominal frequency of the network, $i \in \mathcal{V}_I$. The term P_i has the same expression as in (23).

As for nodal dynamics corresponding to the loads, we consider frequency dependent loads given by the first-order system

$$A_i \dot{\theta}_i = \delta_i - P_i, \quad i \in \mathcal{V}_L \quad (25)$$

Again, here $\dot{\theta}_i$ is the frequency deviation, $A_i > 0$ is the damping coefficient, P_i is given by (23), and δ_i is the constant power consumption at node $i \in \mathcal{V}_L$.

To write the system in a compact form, we need the following nomenclature. For each $i = 1, 2, \dots, k$, let γ_k be defined as $\gamma_k = (\text{Im } Y_{ij}) V_i V_j$ with $\{i, j\}$ being the k^{th} edge of the graph, where the edge numbers are in accordance with the incidence matrix B . We define the diagonal matrix Γ as $\Gamma = \text{diag}(\gamma_j)$, with $j = 1, 2, \dots, k$. Let the matrices B_G , B_I , and B_L be obtained from B by collecting the rows indexed by \mathcal{V}_G , \mathcal{V}_I , and \mathcal{V}_L , respectively. We define the vectors and matrices M_G , A_G , θ_G , and u_G , as $M_G = \text{diag}(M_i)$, $A_G = \text{diag}(A_i)$, $\theta_G = \text{col}(\theta_i)$, $u_G = \text{col}(u_i)$, and $\delta_G = \text{col}(\delta_i)$ where $i \in \mathcal{V}_G$. The vectors and matrices A_I , θ_I , and u_I are defined as $A_I = \text{diag}(A_i)$, $\theta_I = \text{col}(\theta_i)$, $u_I = \text{col}(u_i)$, and $\delta_I = \text{col}(\delta_i)$ with $i \in \mathcal{V}_I$. In addition, let $A_L = \text{diag}(A_i)$, $\theta_L = \text{col}(\theta_i)$ and $\delta_L = \text{col}(\delta_i)$ where $i \in \mathcal{V}_L$. Finally, let $P = \text{col}(P_i)$, $\theta = \text{col}(\theta_G, \theta_I, \theta_L)$, and $\underline{\sin}(x) := \text{col}(\sin(x_i))$ for a given vector x . Then, it is easy to observe that the dynamics of the synchronous generators, the inverters, and the loads can be written compactly as:

$$M_G \ddot{\theta}_G + A_G \dot{\theta}_G = -B_G \Gamma \underline{\sin}(B^T \theta) + u_G - \delta_G \quad (26a)$$

$$A_I \dot{\theta}_I = -B_I \Gamma \underline{\sin}(B^T \theta) + u_I - \delta_I \quad (26b)$$

$$A_L \dot{\theta}_L = -B_L \Gamma \underline{\sin}(B^T \theta) + \delta_L \quad (26c)$$

Note that this is the same model as [8], see also [19]. By defining $\eta = B^T \theta$, $\omega_G = \dot{\theta}_G$, $\omega_I = \dot{\theta}_I$, $\omega_L = \dot{\theta}_L$, and $\dot{\theta} = \omega = \text{col}(\omega_G, \omega_I, \omega_L)$, the network dynamics (26), admits the following representation

$$\dot{\eta} = B^T \omega \quad (27a)$$

$$M_G \dot{\omega}_G + A_G \omega_G = -B_G \Gamma \underline{\sin}(\eta) + u_G + \delta_G \quad (27b)$$

$$A_I \omega_I = -B_I \Gamma \underline{\sin}(\eta) + u_I + \delta_I \quad (27c)$$

$$A_L \omega_L = -B_L \Gamma \underline{\sin}(\eta) + \delta_L \quad (27d)$$

Now, let $p_G = M_G \omega_G$, $H_G = \frac{1}{2} p_G^T M_G^{-1} p_G$, $H_I = \frac{1}{2} \omega_I^T \omega_I$, $H_L = \frac{1}{2} \omega_L^T \omega_L$, and $H_e = -\mathbf{1}^T \Gamma \underline{\cos}(\eta)$. Then, (27) can be written as

$$\dot{\eta} = B^T \nabla H_T(p) \quad (28a)$$

$$\dot{p}_G = -A_G \nabla H_G(p_G) - B_G \nabla H_e(\eta) + u_G + \delta_G \quad (28b)$$

$$0 = -A_I \nabla H_I(\omega_I) - B_I \nabla H_e(\eta) + u_I + \delta_I \quad (28c)$$

$$0 = -A_L \nabla H_L(\omega_L) - B_L \nabla H_e(\eta) + \delta_L \quad (28d)$$

where $p = \text{col}(p_G, \omega_I, \omega_L)$ and $H_T = H_G + H_I + H_L$. Note that (28) has a similar structure/properties as (12), with $\Omega_n = \mathbb{R}$, $\Omega_e = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\mathcal{I}_{12} = \emptyset$. The primary control goal here is to achieve a zero frequency deviation for the power network. As $\nabla H_T = w$, this is in accordance with our definition of output agreement with $y^* = 0$. Moreover, we would like to achieve an optimal steady-state distribution of the power in the sense of (18). In this case, (18) reads as

$$\bar{u}_i = q_i^{-1} \lambda \quad (29)$$

where

$$\lambda = -\left(\sum_i q_i\right)^{-1} (\mathbf{1}^T \delta_G + \mathbf{1}^T \delta_I + \mathbf{1}^T \delta_L).$$

Observe that the feasibility condition (20) in this case amounts for the existence of a constant vector $\bar{\eta} \in (-\frac{\pi}{2}, \frac{\pi}{2})^M$ such that

$$0 = -B_G \nabla H_e(\bar{\eta}) + \bar{u}_G + \delta_G \quad (30a)$$

$$0 = -B_I \nabla H_e(\bar{\eta}) + \bar{u}_I + \delta_I \quad (30b)$$

$$0 = -B_L \nabla H_e(\bar{\eta}) + \delta_L \quad (30c)$$

where \bar{u}_i is given by (29) for each $i \in \mathcal{V}_G \cup \mathcal{V}_I$. Now, assume that the feasibility condition (30) holds. Then, by Theorem 4, the controller

$$\dot{\xi}_i = \sum_{\{i,j\} \in \mathcal{E}_c} (\xi_j - \xi_i) - q_i^{-1} \omega_i \quad (31a)$$

$$u_i = q_i^{-1} \xi_i, \quad i \in \mathcal{V}_G \cup \mathcal{V}_I \quad (31b)$$

achieves zero frequency deviation, and moreover u_i asymptotically converges to the optimal \bar{u}_i given by (29).

Now, consider the case where a proper subset of generators, say $\mathcal{V}_F \subset \mathcal{V}_G$, encounter some failures. In particular, assume that u_i is not appropriately actuated, and is equal to some unknown constant vector for each $i \in \mathcal{V}_F$. Then, for the nodes in the *fail mode*, subdynamics (28b) reads as

$$\dot{p}_F = -A_F \nabla H_F(p_F) - B_F \nabla H_e(\eta) + \delta_F \quad (32)$$

where we have used the index “F” to distinguish the subdynamics above from the nominal subdynamics (28b). Assume that there exists $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})^M$ such that (30) and

$$0 = -B_F \nabla H_e(\bar{\eta}) + \delta_F$$

are satisfied. Note that (30a) has to be modified accordingly to exclude the faulty generators. Observe that (32) has the same structure as (12c). Then, by Theorem 4, we conclude that the controller (31) achieves a zero frequency deviation, and we have optimal steady state distribution of the power, given by (29), despite the failures in the nodal dynamics \mathcal{V}_G .

Note that, similarly, absence or failure of actuation in inverters can be incorporated in our design, as this results in a similar dynamics to that of the loads.

V. CONCLUSIONS

We have investigated the problem of output agreement in heterogeneous networks with port-Hamiltonian nodal dynamics, dynamic physical coupling, and algebraic constraints. We have considered the case where control and disturbance signals may act on different subset of nodes. We have observed that the equilibrium of the network, if exists, is locally attractive, and thus output variables asymptotically converges to a same vector. As discussed, this vector can be steered to a desired one by applying decentralized dynamic controllers at the nodes, upon the satisfaction of certain feasibility conditions imposed by the physics of the problem. We have also studied the case in which we are interested in an optimal steady-state distribution of control signals over the network. As observed, this goal can be achieved by exploiting distributed controllers at the nodes. We have applied the proposed results on a heterogeneous microgrid. Extending the analysis to incorporate time-varying disturbances is a subject of future research.

VI. APPENDIX

Poof of Theorem 1: From (4a), we have

$$\dot{\eta} = (B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1) + (B^2 \otimes I)^T (G^2)^T \nabla H_n^2(x^2) \quad (33)$$

By (4d), we obtain that

$$\begin{aligned} \dot{\eta} &= (B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1) \\ &\quad + (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2 \\ &\quad \cdot ((B^2 \otimes I) \nabla H_e(\eta) - d^2) \end{aligned}$$

Next, we study the asymptotic behavior of the following subdynamics of (4)

$$\begin{aligned} \dot{\eta} &= (B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1) \\ &\quad + (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2 (B^2 \otimes I) \nabla H_e(\eta) \\ &\quad - (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2 d^2 \end{aligned} \quad (34a)$$

$$\begin{aligned} \dot{x}^{(1)} &= (J^{(1)} - R^{(1)}) \nabla H_n^{(1)}(x^{(1)}) \\ &\quad - G^1 (B^{(1)} \otimes I) \nabla H_e(\eta) + G^{(1)} d^{(1)} \end{aligned} \quad (34b)$$

Let W_n and W_e be defined as

$$W_n(x^1, \bar{x}^1) = H_n^1(x^1) - H_n^1(\bar{x}^1) - (\nabla H_n^1(\bar{x}^1))^T (x^1 - \bar{x}^1) \quad (35)$$

and

$$W_e(\eta, \bar{\eta}) = H_e(\eta) - H_e(\bar{\eta}) - (\nabla H_e(\bar{\eta}))^T (\eta - \bar{\eta}) \quad (36)$$

where $(\bar{x}^1, \bar{\eta})$ is an equilibrium of (34). Following [9], W_n identifies a positive definite map with a strict local minimum at $x^1 = \bar{x}^1$. Also W_e defines a positive definite map with a strict local minimum at $\eta = \bar{\eta}$. Noting that $\dot{x} = 0$, we have

$$\begin{aligned} \dot{W}_n &= (\nabla H_n^1(x^1))^T \dot{x}^1 - (\nabla H_n^1(\bar{x}^1))^T (x^1 - \bar{x}^1) \\ &= (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T (x^1 - \bar{x}^1) \\ &= (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T \\ &\quad \cdot (J^1 - R^1) (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1)) \\ &\quad - (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T \\ &\quad \cdot G^1 (B^1 \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned}$$

In addition, noting that $\dot{\eta} = 0$ we have

$$\begin{aligned} \dot{W}_e &= (\nabla H_e(\eta))^T \dot{\eta} - (\nabla H_e(\bar{\eta}))^T (\eta - \bar{\eta}) \\ &= (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (\eta - \bar{\eta}) \\ &= (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^1 \otimes I)^T \\ &\quad \cdot (G^1)^T (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1)) \\ &\quad + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^2 \otimes I)^T (G^2)^T \\ &\quad \cdot (J^2 - R^2)^{-1} G^2 (B^2 \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \quad (37)$$

Let $W_T := W_n + W_e$. Then, we have

$$\begin{aligned} \dot{W}_T &= (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T (J^1 - R^1) \\ &\quad \cdot (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1)) \\ &\quad + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^2 \otimes I)^T (G^2)^T \\ &\quad \cdot (J^2 - R^2)^{-1} G^2 (B^2 \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned}$$

where we have used the fact that d^1 and d^2 are constant.

Now, note that for any skew-symmetric matrix J and a positive definite matrix R , we have $-2R = (J - R) + (J - R)^T < 0$, and thus $(J - R)^{-1} + (J - R)^{-T} < 0$. Hence, $z^T(J - R)z < 0$ and $z^T(J - R)^{-1}z < 0$ for any nonzero vector z . Therefore, we conclude that $\dot{W}_T \leq 0$.

Observe that W_T has a strict local minimum at $x = \bar{x}^1$ and $\eta = \bar{\eta}$, and hence one can construct a compact level set $\Omega_c \subseteq (\Omega_n)^{|Z_1|} \times (\Omega_e)^M$ around $(\bar{x}^1, \bar{\eta})$ which is forward invariant. This implies that on the interval of definition of a solution to system (4), the variables x^1 and η are bounded. Hence, by (4d), the variables $\nabla H_n^2(x^2)$ are also bounded, and a solution to (4) exists for all t .

Then by invoking LaSalle invariance principle, on the invariant set $\dot{W}_T = 0$, we have

$$\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1) = 0 \quad (38a)$$

$$G^2(B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) = 0. \quad (38b)$$

Due to the strict convexity of H_n^1 , (38a) yields $x^1 = \bar{x}^1$. Besides, (34a) admits the following incremental model

$$\begin{aligned} \dot{\eta} = & (B^1 \otimes I)^T (G^1)^T (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1)) \\ & + (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} \\ & \cdot G^2(B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned}$$

Therefore, by (38), we obtain that $\eta = \bar{\eta}$ on the invariant set for some constant vector $\bar{\eta}$, and thus output agreement (6) holds. Note that, by (4d), x_2 asymptotically converges to a constant vector identified by

$$\nabla H_n^2(\bar{x}^2) = (J^2 - R^2)^{-1} G^2((B^2 \otimes I) \nabla H_e(\bar{\eta}) + d^2) \quad (39)$$

This completes the proof. \blacksquare

Proof of Theorem 2: By the algebraic equation (12d), the controller (15) can be written as

$$\dot{\xi}^{11} = \mathbb{1} \otimes y^* - (G^{11})^T \nabla H^{11}(x^{11}) \quad (40a)$$

$$\begin{aligned} \dot{\xi}^{21} = & \mathbb{1} \otimes y^* - (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ & \cdot ((B^{21} \otimes I) \nabla H_e(\eta) - \xi^{21} - \delta^{21}) \end{aligned} \quad (40b)$$

$$u^{11} = \xi^{11} \quad (40c)$$

$$u^{21} = \xi^{21}. \quad (40d)$$

Moreover, by (12a), (12d), and (12e), we have

$$\begin{aligned} \dot{\eta} = & (B^{11} \otimes I)^T (G^{11})^T \nabla H^{11}(x^{11}) \\ & + (B^{12} \otimes I)^T (G^{12})^T \nabla H^{12}(x^{12}) \\ & + (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ & \cdot ((B^{21} \otimes I) \nabla H_e(\eta) - \xi^{21} - \delta^{21}) \\ & + (B^{22} \otimes I)^T (G^{22})^T (J^{22} - R^{22})^{-1} G^{22} \\ & \cdot ((B^{22} \otimes I) \nabla H_e(\eta) - \delta^{22}) \end{aligned} \quad (41)$$

The equation above together with (12b), (12c), and (40) defines a dynamical system with ordinary differential equations,

the solution of which exists and is unique. Moreover, this system admits the following incremental model.

$$\begin{aligned} \dot{\eta} - \dot{\bar{\eta}} = & (B^{11} \otimes I)^T (G^{11})^T (\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11})) \\ & + (B^{12} \otimes I)^T (G^{12})^T \nabla (H^{12}(x^{12}) - H^{12}(\bar{x}^{12})) \\ & + (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ & \cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ & - (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} (\xi^{21} - \bar{\xi}^{21}) \\ & + (B^{22} \otimes I)^T (G^{22})^T (J^{22} - R^{22})^{-1} G^{22} \\ & \cdot (B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \quad (42a)$$

$$\begin{aligned} \dot{x}^{11} - \dot{\bar{x}}^{11} = & (J^{11} - R^{11})(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ & - G^{11}(B^{11} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ & + G^{11}(\xi^{11} - \bar{\xi}^{11}) \end{aligned} \quad (42b)$$

$$\begin{aligned} \dot{x}^{12} - \dot{\bar{x}}^{12} = & (J^{12} - R^{12})(\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ & - G^{12}(B^{12} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \quad (42c)$$

$$\dot{\xi}^{11} - \dot{\bar{\xi}}^{11} = - (G^{11})^T (\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11})) \quad (42d)$$

$$\begin{aligned} \dot{\xi}^{21} - \dot{\bar{\xi}}^{21} = & - (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ & \cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ & + (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} (\xi^{21} - \bar{\xi}^{21}) \end{aligned} \quad (42e)$$

where $\bar{\xi}^{11} = d^{11} - \delta^{11}$, $\bar{\xi}^{21} = d^{21} - \delta^{21}$, and constant vectors \bar{x} and $\bar{\eta}$ are such that (14) is satisfied. Note that, due to the feasibility condition, $(\bar{x}, \bar{\eta}, \bar{u})$ is a valid solution for (12), where $\bar{u} = \text{col}(\bar{u}^{11}, \bar{u}^{21})$, $\bar{u}^{11} = \bar{\xi}^{11}$, and $\bar{u}^{21} = \bar{\xi}^{21}$.

Now consider the Lyapunov function $V = W_n + W_e + W_c$ where

$$W_c = \frac{1}{2}(\xi^{11} - \bar{\xi}^{11})^T (\xi^{11} - \bar{\xi}^{11}) + \frac{1}{2}(\xi^{21} - \bar{\xi}^{21})^T (\xi^{21} - \bar{\xi}^{21}) \quad (43)$$

and W_n and W_e are given by (35) and (36), respectively. By the use of incremental model (42), we obtain

$$\begin{aligned} \dot{W}_n = & (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (\dot{x}^{11} - \dot{\bar{x}}^{11}) \\ & + (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T (\dot{x}^{12} - \dot{\bar{x}}^{12}) \\ = & (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (J^{11} - R^{11}) \\ & \cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ & - (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (G^{11}) \\ & \cdot (B^{11} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ & + (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T G^{11} (\xi^{11} - \bar{\xi}^{11}) \\ & + (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T (J^{12} - R^{12}) \\ & \cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ & - (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T G^{12} \\ & \cdot (B^{12} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \quad (44)$$

and

$$\begin{aligned}
\dot{W}_e = & (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{11} \otimes I)^T \\
& \cdot (G^{11})^T (\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11})) \\
& + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{12} \otimes I)^T \\
& \cdot (G^{12})^T (\nabla H^{12}(x^{12}) - \nabla H^{12}(\bar{x}^{12})) \\
& + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{21} \otimes I)^T (G^{21})^T \\
& \cdot (J^{21} - R^{21})^{-1} G^{21} (B^{21} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& - (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{21} \otimes I)^T (G^{21})^T \\
& \cdot (J^{21} - R^{21})^{-1} G^{21} (\xi^{21} - \bar{\xi}^{21}) \\
& + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{22} \otimes I)^T (G^{22})^T \\
& \cdot (J^{22} - R^{22})^{-1} G^{22} (B^{22} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))
\end{aligned} \tag{45}$$

Then, \dot{W}_c is computed as

$$\begin{aligned}
\dot{W}_c = & -(\xi^{11} - \bar{\xi}^{11})^T (G^{11})^T (\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11})) \\
& -(\xi^{21} - \bar{\xi}^{21})^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\
& \cdot (B^{21} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& +(\xi^{21} - \bar{\xi}^{21})^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} (\xi^{21} - \bar{\xi}^{21})
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\dot{V} = & \dot{W}_n + \dot{W}_e + \dot{W}_c \\
= & (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (J^{11} - R^{11}) \\
& \cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\
& + (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T (J^{12} - R^{12}) \\
& \cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\
& + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{22} \otimes I)^T (G^{22})^T \\
& \cdot (J^{22} - R^{22})^{-1} G^{22} (B^{22} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& + z^T (J^{21} - R^{21})^{-1} z
\end{aligned}$$

where

$$z = G^{21} (B^{21} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - G^{21} (\xi^{21} - \bar{\xi}^{21})$$

Consequently, $\dot{V} \leq 0$. Note that

$$(x^{11}, x^{12}, \eta, \xi^{11}, \xi^{21}) = (\bar{x}^{11}, \bar{x}^{12}, \bar{\eta}, \bar{\xi}^{11}, \bar{\xi}^{21})$$

is a strict local minimum of V , and thus one can construct a compact level set around this point which is forward invariant. This implies that on the interval of definition of a solution to system (4), the variables x^{11} , x^{12} , η , ξ^{11} , and ξ^{21} are bounded. Therefore, by (12d) and (12e), the variables $\nabla H_n^{21}(x^{21})$, $\nabla H_n^{22}(x^{21})$ are also bounded, and a solution to (12) exists for all t . Now by invoking LaSalle invariance principle, one the invariant set $\dot{V} = 0$, we have

$$\begin{aligned}
\nabla H^{11}(x^{11}) &= \nabla H^{11}(\bar{x}^{11}), \nabla H^{12}(x^{12}) = \nabla H^{12}(\bar{x}^{12}), \\
G^{22}(B^{22} \otimes I) &(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) = 0
\end{aligned}$$

and

$$G^{21} (B^{21} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - G^{21} (\xi^{21} - \bar{\xi}^{21}) = 0$$

Hence, by (42a), we obtain that $\dot{\eta} = \dot{\bar{\eta}} = 0$. In addition, by (42d), we have $\dot{\xi} = \dot{\bar{\xi}} = 0$. Consequently, by (40a), we obtain

that $(G^{11})^T \nabla H^{11}(x^{11}) = (G^{11})^T \nabla H^{11}(\bar{x}^{11}) = \mathbb{1} \otimes y^*$ on the invariant set. This together with $\dot{\eta} = 0$ implies that $y_i = y^*$ for each $i \in \mathcal{V}$. Again note that, by (12d)-(12e), x^{21} and x^{22} asymptotically converge to constant vectors \bar{x}^{21} and \bar{x}^{22} with similar expressions as in (39), where the superscripts are modified accordingly. This completes the proof. \blacksquare

Proof of Theorem 4: The controller (21) can be written in compact as

$$\begin{bmatrix} \dot{\xi}^{11} \\ \dot{\xi}^{21} \end{bmatrix} = - (L_c \otimes I) \begin{bmatrix} \xi^{11} \\ \xi^{21} \end{bmatrix} + Q^{-1} \begin{bmatrix} \mathbb{1} \otimes y^* - \nabla H^{11}(x^{11}) \\ \mathbb{1} \otimes y^* - \nabla H^{21}(x^{21}) \end{bmatrix} \tag{46a}$$

$$\begin{bmatrix} u^{11} \\ u^{21} \end{bmatrix} = Q^{-1} \begin{bmatrix} \xi^{11} \\ \xi^{21} \end{bmatrix} \tag{46b}$$

where L_c denotes the Laplacian matrix of \mathcal{G}_c , $Q = \text{blockdiag}(Q_i)$ with $i \in \mathcal{I}_c = \mathcal{I}_{11} \cup \mathcal{I}_{21}$, and in this case

$$\begin{aligned}
\nabla H^{21}(x^{21}) = & (J^{21} - R^{21})^{-1} \\
& \cdot ((B^{21} \otimes I) \nabla H_e(\eta) - (Q^{21})^{-1} \xi^{21} - \delta^{21})
\end{aligned}$$

The controller above admits the following incremental model

$$\begin{aligned}
\begin{bmatrix} \dot{\xi}^{11} - \dot{\bar{\xi}}^{11} \\ \dot{\xi}^{21} - \dot{\bar{\xi}}^{21} \end{bmatrix} = & - (L_c \otimes I) \begin{bmatrix} \xi^{11} - \bar{\xi}^{11} \\ \xi^{21} - \bar{\xi}^{21} \end{bmatrix} \\
& - Q^{-1} \begin{bmatrix} \nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11}) \\ \nabla H^{21}(x^{21}) - \nabla H^{21}(\bar{x}^{21}) \end{bmatrix}
\end{aligned} \tag{47a}$$

$$\begin{bmatrix} u^{11} - \bar{u}^{11} \\ u^{21} - \bar{u}^{21} \end{bmatrix} = Q^{-1} \begin{bmatrix} \xi^{11} - \bar{\xi}^{11} \\ \xi^{21} - \bar{\xi}^{21} \end{bmatrix} \tag{47b}$$

where

$$\begin{aligned}
\nabla H^{21}(x^{21}) - \nabla H^{21}(\bar{x}^{21}) = & (J^{21} - R^{21})^{-1} (B^{21} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& - (J^{21} - R^{21})^{-1} (Q^{21})^{-1} (\xi^{21} - \bar{\xi}^{21})
\end{aligned}$$

The incremental system dynamics in this case is given by

$$\begin{aligned}
\dot{\eta} - \dot{\bar{\eta}} = & (B^{11} \otimes I)^T (\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11})) \\
& + (B^{12} \otimes I)^T (\nabla H^{12}(x^{12}) - \nabla H^{12}(\bar{x}^{12})) \\
& + (B^{21} \otimes I)^T (J^{21} - R^{21})^{-1} (B^{21} \otimes I) \\
& \cdot (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& - (B^{21} \otimes I)^T (J^{21} - R^{21})^{-1} (Q^{21})^{-1} (\xi^{21} - \bar{\xi}^{21}) \\
& + (B^{22} \otimes I)^T (J^{22} - R^{22})^{-1} (B^{22} \otimes I) \\
& \cdot (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))
\end{aligned} \tag{48a}$$

$$\begin{aligned}
\dot{x}^{11} - \dot{\bar{x}}^{11} = & (J^{11} - R^{11}) (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\
& - (B^{11} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
& + (Q^{11})^{-1} (\xi^{11} - \bar{\xi}^{11})
\end{aligned} \tag{48b}$$

$$\begin{aligned}
\dot{x}^{12} - \dot{\bar{x}}^{12} = & (J^{12} - R^{12}) (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\
& - (B^{12} \otimes I) (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))
\end{aligned} \tag{48c}$$

Now, let

$$\bar{\xi}^{11} = \mathbb{1} \otimes \lambda, \quad \bar{\xi}^{21} = \mathbb{1} \otimes \lambda \tag{49}$$

where λ is given by (19). By (49), we have $\bar{u}_i = Q_i^{-1}\lambda$ which coincides with \bar{u}_i given by (18). Hence, by (20), it is easy to observe that $(\bar{x}, \bar{\eta}, \bar{\xi})$ defines a valid solution to (12).

Now consider again the Lyapunov function $V = W_n + W_e + W_c$ where W_n , W_e , and W_c are given by (35), (36), and (43), receptively. Then it is straightforward to investigate that

$$\begin{aligned}\dot{V} = & (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (J^{11} - R^{11}) \\ & \cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ & + (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T (J^{12} - R^{12}) \\ & \cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ & + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{22} \otimes I)^T (J^{22} - R^{22})^{-1} \\ & \cdot (B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ & - \tilde{\xi}^T (L_c \otimes I) \tilde{\xi} + z^T (J^{21} - R^{21})^{-1} z\end{aligned}$$

where $\tilde{\xi} = \text{col}(\xi^{11} - \bar{\xi}^{11}, \xi^{21} - \bar{\xi}^{21})$ and

$$z = (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - (Q^{21})^{-1}(\xi^{21} - \bar{\xi}^{21})$$

Hence, we obtain that $\dot{V} \leq 0$. Note that boundedness, existence, and uniqueness of solution is guaranteed in the same vein as in the proof of Theorem 2. Now by constructing a forward invariant compact level set around $(\bar{x}^{11}, \bar{x}^{12}, \bar{\eta}, \bar{\xi}^{11}, \bar{\xi}^{21})$, and invoking LaSalle invariance principle, on the invariant set we have

$$\nabla H^{11}(x^{11}) - \nabla H^{11}(\bar{x}^{11}) = 0 \quad (50a)$$

$$\nabla H^{12}(x^{12}) - \nabla H^{12}(\bar{x}^{12}) = 0 \quad (50b)$$

$$(B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) = 0 \quad (50c)$$

$$(L_c \otimes I)\tilde{\xi} = 0 \quad (50d)$$

$$(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - (Q^{21})^{-1}(\xi^{21} - \bar{\xi}^{21}) = 0 \quad (50e)$$

Therefore, by (48a), we have $\dot{\eta} = \dot{\bar{\eta}} = 0$. Moreover, by (47a), (50a), and (50d), we obtain that $\dot{\xi}^{11} = \dot{\bar{\xi}}^{11} = 0$ on the invariant set. In addition, (50d) implies that $\tilde{\xi} = \mathbb{1} \otimes \alpha$ for some vector α . Replacing this into (46a) yields $\nabla H^{11}(x^{11}) = \nabla H^{11}(\bar{x}^{11}) = \mathbb{1} \otimes y^*$. This together with $\dot{\eta} = 0$ results in $y_i = y^*$ for each $i \in \mathcal{V}$. Note that on the invariant set, $\xi^{11} = \bar{\xi}^{11} + \mathbb{1} \otimes \alpha = \mathbb{1} \otimes (\alpha + \lambda)$. Similarly, we have $\xi^{21} = \mathbb{1} \otimes (\alpha + \lambda)$, and, hence the system dynamics takes the form

$$\begin{aligned}0 = & (J^{11} - R^{11})(\mathbb{1} \otimes y^*) - (B^{11} \otimes I)\nabla H_e(\bar{\eta}) \\ & + (Q^{11})^{-1}\mathbb{1} \otimes (\lambda + \alpha) + \delta^{11} \\ 0 = & (J^{12} - R^{12})(\mathbb{1} \otimes y^*) - (B^{12} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{12} \\ 0 = & (J^{21} - R^{21})(\mathbb{1} \otimes y^*) - (B^{21} \otimes I)\nabla H_e(\bar{\eta}) \\ & + (Q^{21})^{-1}\mathbb{1} \otimes (\lambda + \alpha) + \delta^{21} \\ 0 = & (J^{22} - R^{22})(\mathbb{1} \otimes y^*) - (B^{22} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{22}\end{aligned}$$

where $\bar{\eta}$ is a constant vector. By multiplying each of the equalities above from the left by $\mathbb{1}^T \otimes I$ and taking the sum over all the resulting equalities, we conclude that

$$\lambda + \alpha = -\left(\sum_{i \in \mathcal{I}_c} Q_i^{-1}\right)^{-1} \left(\sum_{i=1}^N (J_i - R_i)y^* + \sum_{i=1}^N \delta_i\right)$$

By comparing the equality above to (19), we obtain that $\alpha = 0$. Consequently, on the invariant set u_i is equal to the optimal \bar{u}_i given by (18). ■

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